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Modal Analysis of Small Frames Using High Order Timoshenko Beams

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Abstract — In this paper, we consider the modal analysis of small frames. Firstly, we construct the 3D model using H8 elements and find the natural frequencies of the frame focusing our attention on the modes in the XY plane. Secondly, we construct the 2D model (plane stress model) using Q4 elements. We concluded that the results of both models are very close to each other's. Then we formulate the stiffness matrix and the mass matrix of the 3-noded Timoshenko beam that is well suited for thick and short beams like in our case. Finally, we model the corners where the horizontal and vertical bar meet with a special matrix. The results of our new model (3-noded Timoshenko beam for the horizontal and vertical bars and a special element for the corners based on the Q4 elements) are very satisfying when performing the modal analysis.

Index Terms — Corner element, Guyan reduction, High-order Timoshenko beam, modal analysis of frames, rigid link, shear locking, and short beams.

Notations

E	Young's modulus of elasticity	v	Poisson coefficient
ρ	density of the material	h	height of the beam cross-section
b	width of the beam cross section	A	area of the beam cross section
I	moment of inertia of the beam cross section	K	stiffness matrix of the element
M	mass matrix of the element	dof	degree-of-freedom

1 Introduction

The Euler-Bernoulli beam element is the most used element for performing the modal analysis of beams and frames. This type of beam element gives an exact solution for the modal analysis problem given that we have long and slender beam. Whether the beam is clamped, pinned, or free, from any side, meshing it with EB elements will produce excellent results. However, when the beam becomes more and more short, i.e., when the ratio of the width of the beam to its length is > 0.1, the EB beam elements are no longer valid for a modal analysis. We must use Timoshenko beam elements for these cases. The 2-noded Timoshenko beam element is very much used in most software and analyses [1]. In the next paragraph, we will explain how the stiffness and mass matrix of such a beam element are calculated using linear simple shape functions. Following the same procedure, but using high-order shape functions, say quadratic ones, we will formulate the stiffness and mass matrix for this new 3-noded element that we will call "Timo3" element.

2 The Regular 2-Noded Timoshenko Beam Element

In a Timoshenko beam theory, plane sections remain plane after deformation but not necessary perpendicular to the neutral axis. The plane section rotates by an amount, θ, equal to the rotation of the neutral axis, μ, minus the shear strain γ.

The strain energy for an element of length L is [2]:

$$U = \frac{b}{2}\int_0^L \int_{-h/2}^{h/2} \varepsilon^T\, E\, \varepsilon\, dy\, dx + \frac{b\mu}{2}\int_0^L \int_{-h/2}^{h/2} \gamma^T\, G\, \gamma\, dy\, dx \qquad (1)$$

Where, L is the length of the element, b and h are the width and the height of the beam respectively. μ is the correction factor for shear energy; generally taken 5/6 for beams with standard rectangular cross sections and 9/10 for circular section beams [3]. Many formulations of the Timoshenko beam exist, [4], [5], [6], and [7].

The degrees of freedom of this element are:

v_1: transverse displacement of the beam at the left node

θ_1: rotation of the beam section at the left node

v_2: transverse displacement of the beam at the right node

θ_2: rotation of the beam section at the right node

In this model, v and θ are independent variables, thus they can be interpolated independently. By using isoparametric linear shape functions for both variables v and θ:

$$N_1 = \frac{1}{2}(1-\xi) \qquad N_2 = \frac{1}{2}(1+\xi)$$

$$v(\xi) = [N_1(\xi) \quad N_2(\xi)][v_1 \quad v_2]^T$$

$$\theta(\xi) = [N_1(\xi) \quad N_2(\xi)][\theta_1 \quad \theta_2]^T$$

The bending strain is [8]:

$$\kappa = \frac{d\theta}{dx} = \frac{d\theta}{d\xi}\frac{d\xi}{dx} = \left[\frac{dN_1}{d\xi}\theta_1 + \frac{dN_2}{d\xi}\theta_2\right]\frac{d\xi}{dx}$$

The transverse shear strain is:

$$\gamma = \frac{dv}{dx} - \theta = \left[\frac{dN_1}{d\xi}v_1 + \frac{dN_2}{d\xi}v_2\right]\frac{d\xi}{dx} - [N_1\theta_1 + N_2\theta_2]$$

With $\frac{d\xi}{dx} = \frac{L}{2}$, $\frac{dN_1}{d\xi} = -\frac{1}{2}$ $\frac{dN_2}{d\xi} = \frac{1}{2}$

We get

$$\kappa = B_b[v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$$

Where $B_b = \left[0 \quad \frac{-1}{L} \quad 0 \quad \frac{1}{L}\right]$ is the bending matrix of the element.

And $\gamma = B_s[v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$

Where $B_s = \left[\frac{-1}{L} \quad \frac{\xi-1}{2} \quad \frac{1}{L} \quad \frac{-\xi-1}{2}\right]$ is the shear strain matrix of the element?

The virtual displacement is dv = N.$[dv_1 \quad d\theta_1 \quad dv_2 \quad d\theta_2]^T$

and the virtual strains are:

$d\kappa = B_b[dv_1 \quad d\theta_1 \quad dv_2 \quad d\theta_2]^T$

$d\gamma = B_s[dv_1 \quad d\theta_1 \quad dv_2 \quad d\theta_2]^T$

The bending moment is

M = D_b.B_b.$[v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$ where D_b = E I

And the shear force is

V = D_s.B_s.$[v_1 \quad \theta_1 \quad v_2 \quad \theta_2]^T$ where D_s = μ G A

The bending stiffness matrix for the element is computed from:

$$K_b = \int_\Omega B_b^T D_b B_b dx$$

The shear stiffness matrix:

$$K_s = \int_\Omega B_s^T D_s B_s dx$$

And the consistent mass matrix is computed from:

$$M = \int_\Omega \rho A N^T N dx$$

Using natural coordinates,

$K_b = \int_{-1}^{1} B_b^T D_b B_b \frac{L}{2} d\xi$

$K_s = \int_{-1}^{1} B_s^T D_s B_s \frac{L}{2} d\xi$

$$M = \int_{-1}^{1} \rho A N^T N \frac{L}{2} d\xi$$

In order to avoid shear locking, K_s is obtained using the reduced integration technique (one order less than required) [9]. Upon integrating, we get:

$$K_b = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & EI/L & 0 & -EI/L \\ 0 & 0 & 0 & 0 \\ 0 & -EI/L & 0 & EI/L \end{bmatrix}$$

$$K_s = \begin{bmatrix} \mu GA/L & \mu GA/2 & -\mu GA/L & \mu GA/2 \\ \mu GA/2 & \mu GAL/4 & -\mu GA/2 & \mu GAL/4 \\ -\mu GA/L & -\mu GA/2 & \mu GA/L & -\mu GA/2 \\ \mu GA/2 & \mu GAL/4 & -\mu GA/2 & \mu GAL/4 \end{bmatrix}$$

$$M = \rho AL/6 \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 HIGH ORDER TIMOSHENKO BEAM ELEMENT (QUADRATIC)

Let us consider the Timoshenko beam element with 3 nodes shown in Figure 1.

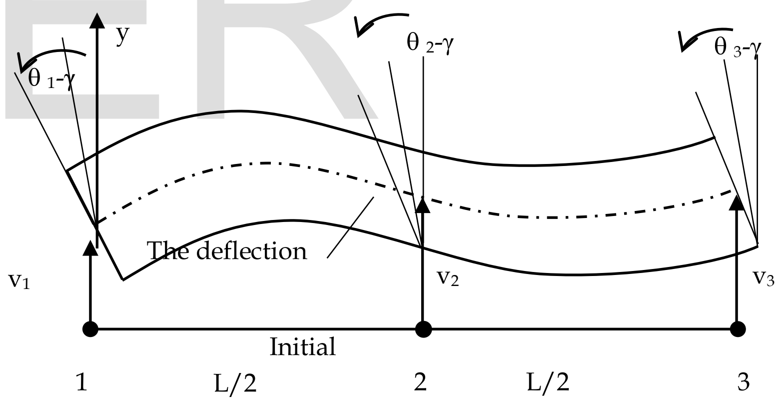

Figure 1. 3-noded Timoshenko beam element

If we use the same procedure as in the previous paragraph, taking 3 nodes per elements, the new Lagrange quadratic shape functions will be:

$$N_1 = \frac{1}{2}(-\xi + \xi^2) \quad N_2 = (1 - \xi^2) \quad N_3 = \frac{1}{2}(\xi + \xi^2)$$

$v(\xi) = N_1(\xi)v_1 + N_2(\xi)v_2 + N_3(\xi)v_3$

$\theta(\xi) = N_1(\xi)\theta_1 + N_2(\xi)\theta_2 + N_3(\xi)\theta_3$

We get

$\kappa = B_b[v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \quad v_3 \quad \theta_3]^T$

Where $B_b = \left[0 \quad \frac{2}{L}\left(\xi - \frac{1}{2}\right) \quad 0 \quad \frac{-4\xi}{L} \quad 0 \quad \frac{2}{L}\left(\xi + \frac{1}{2}\right)\right]$ is the bending matrix of the element.

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And $\gamma = B_s[v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \quad v_3 \quad \theta_3]^T$

Where

$$B_s = \left[\frac{2}{L}\left(\xi - \frac{1}{2}\right) \quad \frac{1}{2}(\xi - \xi^2) \quad \frac{-4\xi}{L} \quad \xi^2 - 1 \quad \frac{2}{L}\left(\xi + \frac{1}{2}\right) \quad \frac{1}{2}(-\xi - \xi^2)\right]$$

is the shear strain matrix of the element.

Again, matrix K_b is obtained using the exact integration whereas K_s is obtained using the reduced integration technique (one order less than required). Upon integrating, we get:

$$K_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & -8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 16 & 0 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -8 & 0 & 7 \end{bmatrix}$$

$$K_b = \mu GA/(36L) \begin{bmatrix} 84 & 18L & -96 & 24L & 12 & -6L \\ 18L & 4L^2 & -24L & 4L^2 & 6L & -2L^2 \\ -96 & -24L & 192 & 0 & -96 & 24L \\ 24L & 4L^2 & 0 & 16 & -24L & 4L^2 \\ 12 & 6L & -96 & -24L & 84 & -18L \\ -6L & -2L^2 & 24L & 4L^2 & -18L & 4L^2 \end{bmatrix}$$

$$M = \rho AL/30 \begin{bmatrix} 4 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 16 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4 TESTING THE 3-NODED TIMOSHENKO BEAM

Let's perform the modal analysis of a clamped-free beam.

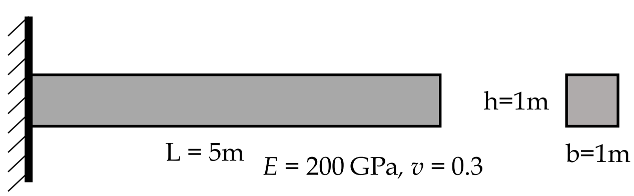
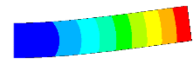
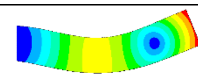
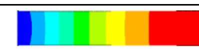
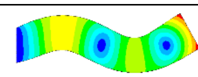
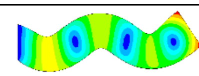
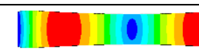


Figure 2. Cantilever beam

The results of the modal analysis of this clamped-free beam are listed in Table 1.

Table 1. Modal analysis of the cantilevered beam for different modelling

Mode No	Mode shape	Q4 Freq (Hz)	EB	Error	Timo3	Error
1		32	33	3%	32	0%
2		172	205	19%	177	3%
3		254	253	0%	253	0%
4		411	574	40%	429	4%
5		684	760	11%	720	5%
6		759	1125	48%	760	0%
	Average			20%		2%

As we can see, since our beam is short (length = 5 m, width = 1 m), the Timo3 beam element shows better performance even in small number of mesh elements (10).

5 MODAL ANALYSIS OF A SMALL L-FRAME

Let's test the Timo3 beam element on a small L-frame clamped at its bottom as shown in the next figure:

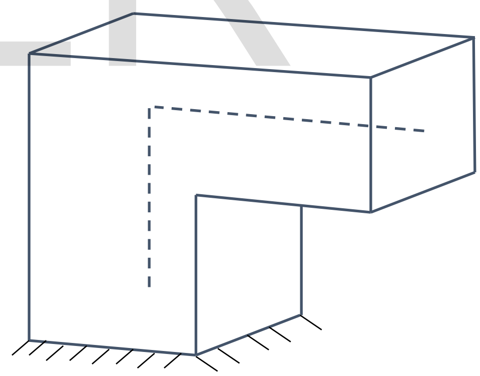
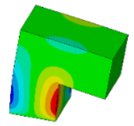
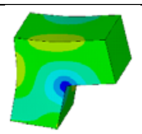
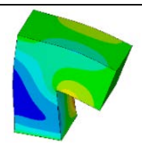
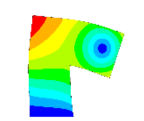
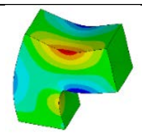
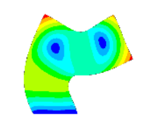
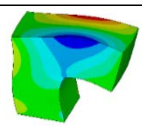
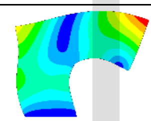
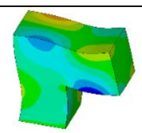
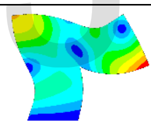


Figure 3. Small L-frame test

All beams have rectangular sections (2x2 m²).
Poisson coefficient v = 0.3; Density ρ = 7800 Kg/m³;
Elastic modulus E = 200 GPa.

Table 2 shows the results of the modal analysis of the L-frame using 2 types of meshing: the first type is the volumetric meshing with H8 elements and the second type is the surface meshing (plane stress condition) with Q4 elements.

Table 2. Modal analysis of the small L-frame

Mode No	Mode shape (3D)	Freq (Hz) H8	Mode shape (2D)	Freq (Hz) Q4
1		55		54
2		148		146
3		296		294
4		455		454
5		582		584
6		655		656

Since the modal analysis of the Q4 model is close enough to the results of the H8 model, we will drop the volumetric H8 meshing and focus our attention on the 2D meshing with Q4 elements. In the next paragraphs, our reference model will be the Q4 model.

6 MODELING THE SMALL L-FRAME USING TIMO3 BEAM ELEMENTS

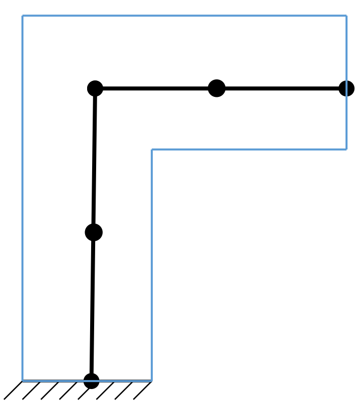
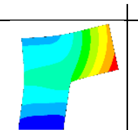
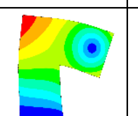
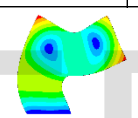
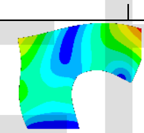
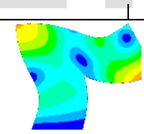


Figure 4. Modeling the test L-frame using Timo3 elements

Let's model our frame using regular EB beam elements or Timo3 beam elements and compare the results of the modal analysis with the reference model the Q4 model.

Table 3 lists the results of the first six mode shapes with their corresponding natural frequencies for different modeling.

Table 3. Modal analysis of the L-frame using beam elements modeling

Mode No	Mode shape	Freq (Hz) Q4	EB	Error	Timo3	Error
1		54	60	11%	56	4%
2		146	151	3%	111	24%
3		294	344	17%	324	10%
4		454	639	41%	418	8%
5		584	853	46%	613	5%
6		656	1089	66%	728	11%
Average				31%		10%

As we can see from the previous table, the Timo3 model gives closer results than the EB model when compared to the results from the reference model the Q4 model. Still, the average error of the first six mode shapes of the Timo3 is 10.3% in comparison to the reference model the Q4 model. That shows that we need to make some further adjustments to our linear model in order to decrease that error. In the next paragraph, we will explain the method used that will reduce the error.

7 MODELING THE CONNECTION

The bars of our test L-frame are modeled using Timo3 beam elements, but the connection that is large in our small L-frame must be given special consideration. We propose a special element for that connection based on the utilization of Q4 surface elements. There are many ways in the finite element method to connect two different element types like in [10], [11], and [12].

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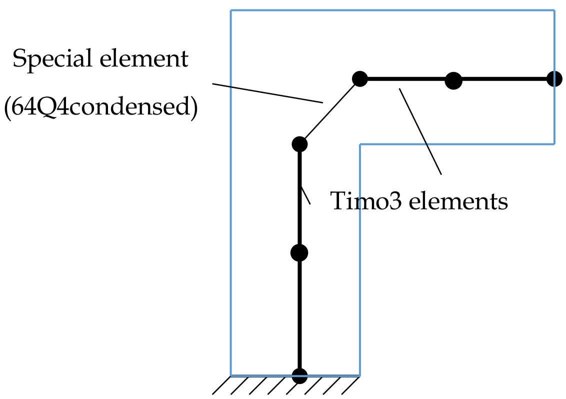


Figure 5. Special treatment for the connection (at the corner)

The interface between the new "64Q4condensed" element and the Timo3 elements (on the vertical and the horizontal bars) is considered a rigid link. The corner element of 2 m width and 2 m height is meshed using 64 Q4 elements as shown in the next Figure.

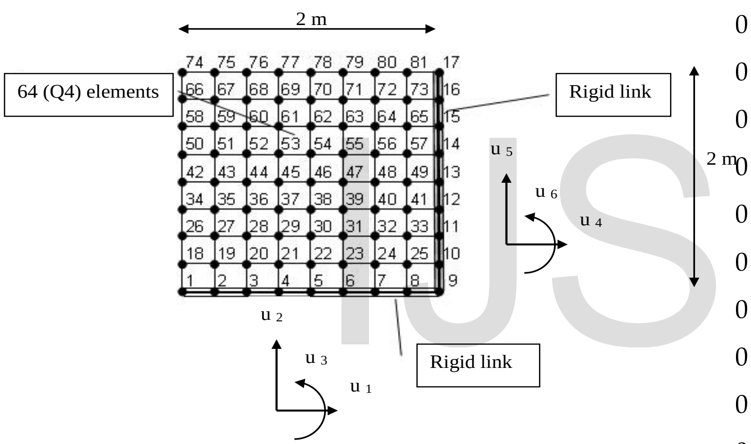


Figure 6. Corner modeled with 64 Q4 elements and rigid links at the interface

The stiffness matrix $k_e(8\times8)$ and the mass matrix $M_e(8\times8)$ of each Q4 element are known [13]. We assemble the 64 Q4 elements to find the global stiffness and mass matrix of the corner $k(162\times162)$ and $M(162\times162)$. Using static condensation (Guyan reduction technique), we can reduce the global stiffness matrices to $k_c(34\times34)$ and $M_c(34\times34)$ after elimination of the 128 non-interface dof. The rigid link can only translate in 2 directions and rotate in one direction thus 3 dof are needed at each interface. Therefore, our special element will have 6 dof, 3 at each node.

The reduced stiffness matrix $k_r(6\times6)$ is given by: $k_r = L^T k_c L$,

Where L(34×6) is the matrix giving the values of the dof relative to both interfaces given by:

L=[

1	0	0	0	0	0
0	1	-4*a	0	0	0
1	0	0	0	0	0
0	1	-3*a	0	0	0
1	0	0	0	0	0
0	1	-2*a	0	0	0
1	0	0	0	0	0
0	1	-1*a	0	0	0
1	0	0	0	0	0
0	1	0	0	0	0
1	0	0	0	0	0
0	1	1*a	0	0	0
1	0	0	0	0	0
0	1	2*a	0	0	0
1	0	0	0	0	0
0	1	3*a	0	0	0
0	0	0	1	0	4*b
0	1	4*a	0	0	0
0	0	0	1	0	3*b
0	0	0	0	1	0
0	0	0	1	0	2*b
0	0	0	0	1	0
0	0	0	1	0	1*b
0	0	0	0	1	0
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	1	0	-1*b
0	0	0	0	1	0
0	0	0	1	0	-2*b
0	0	0	0	1	0
0	0	0	1	0	-3*b
0	0	0	0	1	0
0	0	0	1	0	-4*b
0	0	0	0	1	0];

Where *a* and *b* are the width and column of the corner divided by 8 respectively.

Using partition matrix notation,

k_{rr} = kk(1:34,1:34); k_{rc} = kk(1:34,35:162);

k_{cr} = kk(35:162,1:34); k_{cc} = kk(35:162,35:162);

$kk_{cond} = k_{rr} - k_{rc} \cdot (k_{cc})^{-1} . k_{cr}$;

Therefore, we can find that $k = L^T.kk_{cond}.L$. Where k(6×6) was found using Matlab®.

The same procedure can be used to find the mass matrix M, keeping in mind that it is a function of the elementary mass matrix of each Q4 element and the stiffness matrix k_{cc} [14].

M_{rr} = MM(1:34,1:34); M_{rc} = MM(1:34,35:162);

M_{cr} = MM(35:162,1:34); M_{cc} = MM(35:162,35:162);

$MM_{cond} = M_{rr} - M_{rc}.(kcc)^{-1}.k_{rc}^{T} - k_{rc}.(kcc)^{-1}.M_{rc}^{T} + k_{rc}.(k_{cc})^{-1}.M_{cc}.(k_{cc})^{-1}.k_{rc}^{T}$;

$M=L^T.MM_{cond}.L$

For our L-frame example, *a* = 2/8, *b* = 2/8, *E*=2e11, *v* = 0.3, *t* = 2, the matrix k for the corner element (64Q4condensed) is

k = [

5.0708e+11 1.6119e+11 -5.9359e+10 -5.0708e+11 -1.6119e+11 -2.8653e+11

1.6119e+11 5.0708e+11 2.8653e+11 -1.6119e+11 -5.0708e+11 5.9359e+10

-5.9359e+10 2.8653e+11 4.062e+11 5.9359e+10 -2.8653e+11 -6.0307e+10

-5.0708e+11 -1.6119e+11 5.9359e+10 5.0708e+11 1.6119e+11 2.8653e+11

-1.6119e+11 -5.0708e+11 -2.8653e+11 1.6119e+11 5.0708e+11 -5.9359e+10

-2.8653e+11 5.9359e+10 -6.0307e+10 2.8653e+11 -5.9359e+10 4.062e+11]

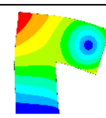
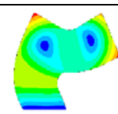
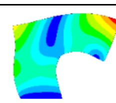
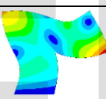
m = [

10588	-2319.7	-1081	7093.5	1524.1	288.55
-2319.7	37625	-8288.2	3115.4	7093.5	-7805.7
-1081	-8288.2	7164.2	-7805.7	288.55	3108
7093.5	3115.4	-7805.7	37625	-2319.7	-8288.2
1524.1	7093.5	288.55	-2319.7	10588	-1081
288.55	-7805.7	3108	-8288.2	-1081	7164.2]

Now that we have found the stiffness matrix k(6,6) and the mass matrix M(6,6) of the corner element; the "64Q4condensed" element with Timo3 elements for the horizontal and vertical bars, we can model our test L-frame and perform the modal analysis.

Table 4 shows the results of the modal analysis of our test L-frame using 64Q4condensed element at the corner and Timo3 beam elements for the horizontal and vertical bars.

Table 4. Modal analysis of the L-frame using improved modeling

Mode No	Shape	Freq (Hz) Q4	Timo3 Error		64Q4condensed Error	
1		54	56	4%	55	2%
2		146	111	24%	150	2%
3		294	324	10%	311	6%
4		454	418	8%	473	4%
5		584	613	5%	602	3%
6		656	728	11%	700	7%
Average				10%		4%

The above table shows the results of the modal analysis of the test L-frame. By using a 64Q4condensed element at the corners the error of the average of the first 6 natural frequencies of the test L-frame was reduced from 10.3% (using only Timo3 beam elements for the horizontal and vertical bars) to 4.1% (using Timo3 beam elements for the horizontal and vertical bars but with a 64Q4condensed element at the corner).

8 ANALYTICAL EXPRESSION OF THE STIFFNESS AND MASS MATRIX OF THE CORNER ELEMENT

We know that the stiffness matrix k(6×6) and the mass matrix M(6×6) are both a function of: The width of the corner, the height of the corner, the elastic modulus *E* of the material at the corner, the Poisson coefficient *v* of the material at the corner, and the density ρ of the material at the corner.

If we let *sl* = *b*/*a* = the ratio of both the height and the width of the corner, the expression of the stiffness matrix will be a function of: *E*, *v*, *sl*, and *a* only.

We can easily show that the stiffness matrix k can be written as k = *E*. $[k_i]$ where E is the Young's modulus of the material at the corner. Let's find the expression of k_i.

If we vary *v* from 0.1 to 0.9 and *sl* from 1 to 10, and by using the modeling technique (the multiple nonlinear regression) we can find the value of each number of the stiffness matrix using

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the function surface fitting in Matlab. For example, the value of k_{11} is:

k_{11}(*v*,sl) = p00 + p10**v* + p01*sl + p20**v*^2 + p11**v**sl + p02*sl^2 + p30**v*^3 + p21**v*^2*sl + p12**v**sl^2 + p03*sl^3 + p40**v*^4 + p31**v*^3*sl + p22**v*^2*sl^2 + p13**v**sl^3 + p04*sl^4 + p50**v*^5 + p41**v*^4*sl + p32**v*^3*sl^2 + p23**v*^2*sl^3 + p14**v**sl^4 + p05*sl^5

Coefficients (with 95% confidence bounds):

p00 = -1.317 (-1.446, -1.189) p10 = 39.28 (38.14, 40.43) p01 = 0.3133 (0.2174, 0.4091)

p20 = -210.7 (-215.6, -205.9) p11 = -3.461 (-3.781, -3.141) p02 = 0.03668 (0.00061, 0.072)

p30 = 496.4 (486.3, 506.5) p21 = 15.54 (14.89, 16.19) p12 = 0.01262 (-0.044, 0.069)

p03 = -0.00452 (-0.011, 0.0022) p40 = -529.8 (-540, -519.6) p31 = -26.81 (-27.5, -26.12)

p22 = -0.01135 (-0.068, 0.045) p13 = -0.00116 (-0.0065, 0.0042) p04 = 0.0003 (-0.0003, 0.0009)

p50 = 209.7 (205.7, 213.7) p41 = 16.33 (16.02, 16.64) p32 = 0.0568 (0.0299, 0.0837)

p23 = -0.0031 (-0.0055, -0.00076) p14 = 0.0001 (-7.3e-5, 0.0003) p05 = -1.07e-5 (-3.3e-5, 1.1e-5)

Goodness of fit: SSE: 39.05 R-square: 0.9986 Adjusted R-square: 0.9986 RMSE: 0.07289

For the mass matrix:

M_{11}(*v*,*sl*) = p00 + p10**v* + p01**sl* + p20**v*^2 + p11**v***sl* + p02**sl*^2 + p30**v*^3 + p21**v*^2**sl* + p12**v***sl*^2 + p03**sl*^3 + p40**v*^4 + p31**v*^3**sl* + p22**v*^2**sl*^2 + p13**v***sl*^3 + p04**sl*^4 + p50**v*^5 + p41**v*^4**sl* + p32**v*^3**sl*^2 + p23**v*^2**sl*^3 + p14**v***sl*^4 + p05**sl*^5

Coefficients (with 95% confidence bounds):

p00 = 13.45 (13.41, 13.5) p10 = -5.285 (-5.661, -4.909) p01 = -0.750 (-0.781, -0.718)

p20 = -20.78 (-22.36, -19.21) p11 = 0.551 (0.446, 0.656) p02 = 0.4257 (0.4139, 0.4376)

p30 = 57.59 (54.3, 60.89) p21 = 0.1597 (-0.05333, 0.3728) p12 = -0.0226 (-0.041, -0.0039)

p03 = -0.0478 (-0.0500, -0.0455) p40 = -57.38 (-60.72, -54.04) p31 = -3.889 (-4.115, -3.663)

p22 = 0.3484 (0.3298, 0.367) p13 = -0.01205 (-0.0138, -0.0102) p04 = 0.0034 (0.0032, 0.0037)

p50 = 21.83 (20.52, 23.15) p41 = 2.482 (2.381, 2.584) p32 = -0.02902 (-0.0378, -0.02)

p23 = -0.016 (-0.0168, -0.0152) p14 = 0.00089 (0.00082, 0.00096) p05 = -0.00011 (-1e-4, -1e-4)

Goodness of fit: SSE: 4.199 R-square: 1 Adjusted R-square: 1 RMSE: 0.0239

The R-square value from all the regression analyses of all terms k_{ij} of the stiffness matrix k and the mass matrix M_{ij} is at least 0.99. Thus, the model fits very well. Using this surface fitting in Matlab, we were able to find the analytical expression of all terms of the stiffness matrix k and mass matrix M of the 64Q4condensed element.

After finding the analytical expression for the stiffness matrix and the mass matrix for the corner element, and by modeling the horizontal and vertical bars with Timo3 elements, let's run the modal analysis on another L-frame. For example:

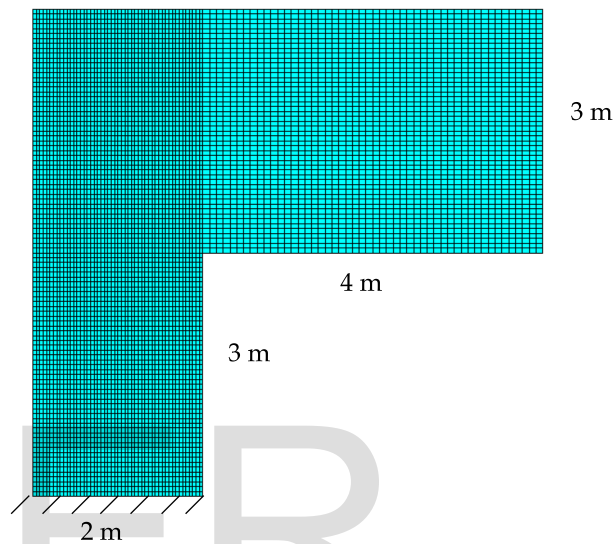
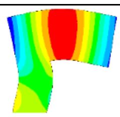
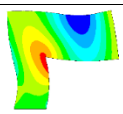
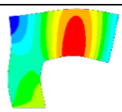


Figure 7. New test L-frame

Running the modal analysis again on this new test L-frame, we get the results shown in Table 5.

Table 5. Modal analysis of another test L-frame using improved modeling

Mode No	Shape	Freq (Hz) Q4	Timo3 Error		64Q4condensed Error	
1		20	20	1%	20	2%
2		66	51	23%	68	2%
3		174	184	6%	183	5%
4		281	258	8%	297	6%
5		371	378	2%	377	2%
6		429	455	6%	453	6%
Average				8%		4%

The previous table shows again that our special modeling at the corner with 64Q4condensed elements gives closer results to the reference model (Q4 model) in modal analysis.

9 CONCLUSION

Replacing the 3D modeling of prismatic frames by traditional linear elements (Timoshenko or Euler-Bernoulli beams) leads to a weak simulation for the dynamical analysis. When the beams are thin and slender, the Euler-Bernoulli beam elements are well suited but they perform poorly in the case of small frames i.e. short beams. By using Timo3 beam elements for the bars and "64Q4condensed" elements at the corner, the model rapidly and easily converges to the exact solution even if we used a small number of elements. First, because our shape functions in the Timo3 beam element are quadratic. Second, the corners are modeled by a 64Q4condensed element that is based on the assembling and condensation of 64 Q4 elements.

Whenever you are running a modal analysis of frames, if the length of beams is short with respect to the width of the cross section of that beam, use the quadratic Timoshenko beam element and give special consideration for the connection at the corner. The best approach is to use Timo3 beam elements for the horizontal and vertical bars and "64Q4condensed" element at the corner.

10 REFERENCES

[1] R. J. Astley, Finite Elements in Solids and Structures : An Introduction, Springer, 1992.

[2] H. Bang and Y. Kwon, he Finite Element Method Using MATLAB, CRC Press, 2000.

[3] S. Timoshenko, Elements of Strength of Materials, D. Van Nostrand Co., 1957.

[4] J. Archer, "Consistent matrix formulations for structural analysis using finite-element techniques," *American Institute of Aeronautics and Astronautics Journal,* pp. 1910-1918, 1965.

[5] J. Thomas and B. Abbas, "Finite element model for dynamic analysis of Timoshenko beam," *Journal of Sound and Vibration,* p. 291–299, 1975.

[6] P. Heyliger and J. Reddy, "A higher order beam finite element for bending and vibration problems," *Journal of Sound and Vibration,* p. 309–326, 1988.

[7] P. Subramanian, "Dynamic analysis of laminated composite beams using higher order theories and finite elements," *Composite Structures,* p. 342–353, 2006.

[8] E. Oñate, Structural Analysis with the Finite Element Method. Linear Statics: Volume 2 Beams, Plates and Shells, Spain: Springer, 2009.

[9] J. Reddy, "On locking-free shear deformable beam finite elements," *Computer Methods Applied Mechanical Engineering,* p. 113–132, 1997.

[10] C. Dohrmann, M. Heinstein and S. Key, "A method for connecting dissimilar finite element meshes in two dimensions," *International Journal For Numerical Methods In Engineering,* pp. 655-678, 2000.

[11] Q. Zu-Qing and C. Wenjun, "Dynamic condensation method for viscously damped vibration systems in engineering," *Engineering Structures,* p. 1426–1432, 2000.

[12] M. Kattner and M. Crisinel, "Finite element modelling of semi-rigid composite joints," *Computers and Structures,* pp. 341-353, 2000.

[13] R. Cook, Concepts and applications of finite element analysis, Wiley, 2002.

[14] T. Chandrupatla and A. Belegundu, Introduction to Finite Elements in Engineering, Prentice Hall, 2011.

[15] D. Thomas, J. Wilson and R. Wilson, "Timoshenko beam finite elements," *Journal of Sound and Vibration,* p. 315–330, 1973.

[16] J. Przemieniecki, Theory of Matrix Structural Analysis, New York: McGraw-Hill, 1968.

[17] G. Cowper, "The Shear Coefficient in Timoshenko's Beam Theory," *Journal of Applied Mechanics,* pp. 335-340, 1966.

[18] "Methods for connecting dissimilar three-dimensional finite element meshes," *International Journal For Numerical Methods In Engineering,* pp. 1057-1080, 2000.